JOURNAL OF COMPUTATIONAL PHYSICS 28, 253–270 (1978)

Variational Methods for Underwater Acoustic Problems*

GEORGE J. FIX AND SAMUEL P. MARIN

Department of Mathematics, Carnegie–Mellon University, Pittsburgh, Pennsylvania 15213

Received September 12, 1977; revised January 9, 1978

Duct acoustic problems differ sharply from the pure exterior problem in that the classical radiation conditions do not prevent wall reflections in the former as they do in the latter. In this paper we derive alternate boundary conditions which do prevent wall reflections, and at the same time can be embedded in a natural way in a Galerkin variational formulation of the duct problem.

1. INTRODUCTION

Wave propagation problems in layers such as arise in underwater acoustics differ rather sharply from pure exterior problems in the types of boundary conditions that are appropriate. In particular, in the layered problem the normal radiation condition can and usually does lead to wall reflections. This creates serious computational problems for standard finite element and finite difference approximations in the sense that both these methods—either by mapping or by truncation—wind up solving a problem in a bounded region. The "conditions at infinity" implied by these techniques invariably lead to reflections. This problem can be avoided by reformulating the boundary value problem as an integral equation [1, 2]; however, the latter has other computational problems such as the need for the exact free space Green's function which often is difficult to obtain in variable coefficient problems.

In this paper we formulate a "generalized radiation condition" that on the one hand prevents wall reflections, and at the same time, can be incorporated in a natural way into a Galerkin type variational formulation of the problem. The latter is used in the present work as a starting point for a finite element scheme, however the variational principle could be used equally well to derive finite difference schemes.

The basic ideas that we use in the variational formulation of layered problems are actually quite general. Marin [3] has shown that they can be effectively applied to exterior problems, and are a natural way to blend standard finite element approximations with integral equation techniques. In addition, Marin has developed a theory for stability and accuracy of the associated approximations.

In concurrent and independent work Engquist and Majda [4] derived generalized

* This work was supported in part by ONR under Contract No. N00014-76-C-0369.

0021-9991/78/0282–0253$02.00/0

generalized outflow boundary conditions using transform techniques. The context for their work was hyperbolic systems in exterior regions, but their results carry over without difficulty to the Helmholtz problem in an exterior region. Our work differs from this approach in that "duct" or "semibounded" regions are treated, and separation of variables rather than transform techniques are used to develop the outflow conditions.

It is perhaps of importance to emphasize that the boundary conditions developed here (as well as the analog in [4]) are *nonlocal.* This is in striking contrast to the classical radiation condition which implies that the normal derivative of the solution ϕ is proportional to ϕ at each point on the boundary. In our context the normal derivative will depend on values of ϕ on the entire exterior boundary.

2. Mathematical Model

To fix ideas we consider an axially symmetric acoustic potential ϕ which satisfies the Helmholtz equation

$$\Delta\phi + \omega^2\phi = 0 \tag{1}$$

in Ω subject to the boundary conditions

$$\phi = g \text{ on } r = r_0, \qquad \phi = 0 \text{ on } z = 0, \qquad \frac{\partial\phi}{\partial n} = 0 \text{ on } z = H(r) \tag{2}$$

(see Fig. I), where z denotes the vertical axis and r denotes the radial coordinate. The

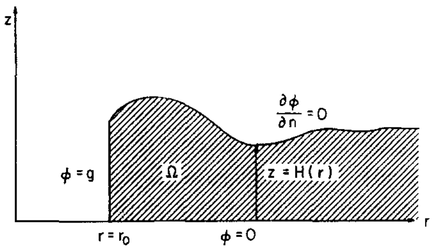

Fig. I. The region Ω.

problem we are addressing is the following. Suppose Ω were truncated at some radius $r = r_\infty$ to give the bounded region

$$\hat{\Omega} = \{(r, z) \in \Omega \mid r_0 < r < r_\infty\}. \tag{3}$$

What types of boundary conditions can be given at $r = r_\infty$ that will prohibit reflections off this boundary?

To get some insight into this problem, and at the same time see that the normal radiation condition

$$\frac{\partial \phi}{\partial r} = i\omega\phi \quad \text{on} \quad r = r_\infty$$

is not always appropriate, let us suppose for the moment that ω and H are constant and

$$\Omega = \{(r, z) | \, r_0 < r, 0 < z < H\}.$$

Following Brekhovskikh [5] a solution exists and has the form

$$\phi = \phi_+ + \phi_-,$$

where $\phi_\pm$ admit the power series

$$\phi_\pm = \sum a_k^\pm \sin \omega_k z H_\pm(\sigma_k r)$$

with

$$\sigma_k^2 = \omega^2 - \omega_k^2, \qquad \omega_k = \frac{(2k-1)\,\pi}{2H},$$

and H_+, H_- are the Hankel functions of the first and second kind of zero order.

We recall that these functions have the asymptotic expansions

$$H_\pm(\xi) = [2/\pi]^{1/2} \exp[\pm i(\xi - \pi/4)]\{\xi^{-1/2} + O(\xi^{3/2})\} \tag{4}$$

as $|\xi| \to \infty$. Thus if

$$\omega < \omega_k, \qquad \text{all } k, \tag{5}$$

$H_+(\sigma_k r)$ is exponentially decreasing to zero as $r \uparrow \infty$ while $H_-(\sigma_k r)$ is exponentially increasing to infinity. The physically correct solution is therefore

$$\phi = \phi_+, \tag{6}$$

where the coefficients $\{a_k^+\}$ are determined from the boundary condition $\phi = g$ at $r = r_0$. Observe that in this case there really is no difficulty with the boundary, and any condition that precludes unbounded solutions is satisfactory.

In the physically more interesting case where (5) fails for some values of k, the situation is completely different. Suppose for the moment that

$$\omega_1 < \omega < \omega_2. \tag{7}$$

If we view ϕ as arising from the wave equation, i.e.,

$$\psi(t) = e^{-i\omega t}\phi,$$

where

$$\frac{\partial^2 \psi}{\partial t^2} = \Delta \psi,$$

then the term

$$H_+(\sigma_1 r) \sin \omega_1 z$$

is associated with a wave moving to the right while

$$H_-(\sigma_1 r) \sin \omega_1 z$$

is associated with a wave moving in the opposite direction. Both of these terms are damped like

$$1/r^{1/2},$$

and the other terms are either decreasing exponentially to zero or are unbounded as $r \uparrow \infty$. Thus the solution that represents only outgoing waves at $r = r_\infty$ is again (6).

The important point to make is that this solution is not determined by the radiation condition (3), and in fact, the latter will yield nonzero values of a_1^-, and hence a reflected wave

$$a_1^- H_-(\sigma_1 r) \sin \omega_1 z$$

will be added to ϕ_+.

Observe that the condition

$$\frac{\partial \phi}{\partial r} = i(\omega^2 - \omega_1^2)^{1/2} \phi \tag{8}$$

(plus boundedness) will generate the correct solution when (7) holds. However, if

$$\omega_1 < \cdots < \omega_j < \omega < \omega_{j+1}$$

for some $j \geqslant 2$, then (8) is no better than (3) in the sense that it will add spurious reflections

$$a_1^- H_-(\sigma_1 r) \sin \omega_1 z + \cdots + a_j^- H_-(\sigma_j r) \sin \omega_j z$$

to the physically correct solution

$$a_1^+ H_+(\sigma_1 r) \sin \omega_1 z + \cdots + a_j^+ H_+(\sigma_j r) \sin \omega_j z.$$

What is needed then is a generalized radiation condition

$$\frac{\partial \phi}{\partial r} = T(\phi) \quad \text{at} \quad r = r_\infty \tag{9}$$

that will prevent reflections in those cases where we cannot solve the partial differential equation exactly (i.e., if ω were not constant or if $\hat{\Omega}$ were not a rectangle).

One way to approach this problem is let ω be variable inside $\hat{\Omega}$ but assume that it is constant on $\Omega - \hat{\Omega}$, and assume that $\Omega - \hat{\Omega}$ is rectangular as in Figure II. In this case the desired solution has the representation

$$\phi(r, z) = \sum_{k=1}^{\infty} a_k H_+(\sigma_k r) \sin \omega_k z \tag{10}$$

for $r \geqslant r_\infty$ where the coefficients a_k are related to the (unknown) values of $\phi(r_\infty, z)$ by

$$a_k = a_k(\phi) = \frac{2}{H_\infty H_+(\sigma_k r_\infty)} \int_0^{H_\infty} dz\ \phi(r_\infty, z) \sin \omega_k z. \tag{11}$$

Thus

$$\frac{\partial \phi}{\partial r} = T(\phi) \quad \text{on} \quad \Gamma = \Gamma_\infty, \tag{12}$$

where

$$T(\phi) = \sum_{k=1}^{\infty} \sigma_k a_k(\phi)\ H_+^\nabla(\sigma_k r) \sin \omega_k z. \tag{13}$$

The boundary value problem is then to solve (1) in $\hat{\Omega}$ subject to

$$\phi = g \text{ on } \Gamma = \Gamma_0;$$

$\partial\phi/\partial n = 0$ on the top wall and $\phi = 0$ on the bottom wall with (12) being the outflow condition at $\Gamma = \Gamma_\infty$.

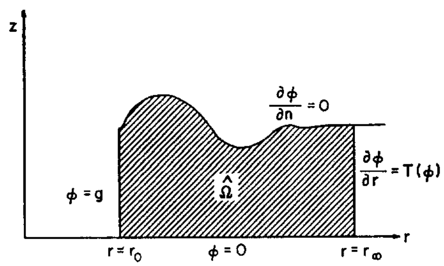

FIG. II. The truncated region Ω.

Observe that the latter is a "nonlocal" boundary condition in the sense that it couples

$$\frac{\partial \phi}{\partial r}(\Gamma_\infty, z)$$

at any point z with the values of $\phi(r_\infty, \cdot)$ at all other points on $r = r_\infty$. However, as we shall show in the next section this does not seriously affect the computational properties of a Galerkin-finite element formulation of this problem.

For high frequency problems where

$$\omega \gg 1, \tag{14}$$

our boundary condition (12) may be very close to the classical radiation condition (3). Indeed, suppose in addition to (14) that ω is not near a critical value ω_k so

$$\sigma_k = \omega(1 - (\omega_k/\omega)^2)^{1/2} \cong \omega$$

for all real σ_k. Then

$$\frac{H_+{}^\nabla(\sigma_k r_\infty)}{H_+(\sigma_k r_\infty)} = i + O(\sigma_k r_\infty)^{-1}.$$

Thus

$$T(\phi) = \sum_k \sigma_k a_k(\phi)\, H_+\mathrm{e}(\sigma_k r) \sin \omega_k z \cong i\omega \sum_k \alpha_k \sin \omega_k z, \tag{15}$$

where

$$\alpha_k = \frac{2}{H_\infty} \int_0^{H_\infty} dy\, \phi(r_\infty, y) \sin \omega_k y.$$

This means that

$$\frac{\partial \phi}{\partial r} \cong i\omega\phi \quad \text{on} \quad r = r_\infty$$

whenever $\sigma_k = (\omega^2 - \omega_k{}^2)^{1/2} \cong \omega$ for those k for which $\sigma_k > 0$. That is, (12) reduces to the standard radiation condition (2) as $\omega \to \infty$.

3. A Variational Formulation

To reformulate our boundary value problem ((1)-(2), (12)) in a variational form we use the classical Galerkin ideas. That is, we multiply (1) by a test function ψ and integrate over $\hat{\Omega}$, to get

$$\int_{\hat{\Omega}} [\Delta\phi + \omega^2\phi]\, \psi = 0.$$

Using Green's theorem on the first term gives

$$\int_{\hat{\Omega}} [-\nabla\phi \cdot \nabla\psi + \omega^2\phi\psi] + \int_{\partial\hat{\Omega}} \frac{\partial\phi}{\partial n}\, \psi = 0.$$

Denoting the line $r = r_\infty$ by Γ_∞ and requiring that ψ satisfy

$$\psi = 0 \text{ on } r = r_0\,,\ \psi = 0 \text{ on } z = 0 \tag{16}$$

we obtain

$$a(\phi, \psi) \equiv \int_{\hat{\Omega}} [-\nabla\phi \cdot \nabla\psi + \omega^2\phi\psi] + \int_{\Gamma_\infty} T(\phi)\,\psi = 0. \tag{17}$$

A precise statement of the variational principle is given below.

PROBLEM VP. Find a function ϕ with square integrable gradient satisfying

$$\phi = g \text{ on } r = r_0\,, \tag{18}$$

$$\phi = 0 \text{ on } z = 0, \tag{19}$$

and for which (17) is true for all ψ (with square integrable gradient) satisfying (16).

Observe that

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on} \quad z = H(z)$$

and

$$\frac{\partial\phi}{\partial n} = T(\phi) \quad \text{on} \quad \Gamma_\infty$$

are natural boundary conditions in this formulation.

To approximate the solution ϕ of this problem we introduce a finite dimensional space S^h and let $S_0{}^h$ be the subspace of functions $\psi \in S^h$ satisfying (16). In addition we select a $g_h \in S^h$ which approximates g on $r = r_0$. Then our approximate problem is the following.

PROBLEM AVP. Find a $\phi_h \in S^h$ such that

$$\phi_h = g_h \text{ on } r = r_0\,, \tag{20}$$

$$\phi_h = 0 \text{ on } z = 0, \tag{21}$$

and such that

$$a(\phi_h\,, \psi^h) = 0 \tag{22}$$

for all $\psi^h \in S_0{}^h$.

As usual this formulation is equivalent to a system of linear equations once a basis $\phi_1{}^h, \ldots, \phi_N{}^h$ has been selected for S^h. Indeed, since $\phi_h - g_h \in S_0{}^h$ we have

$$\phi_h = g_h + \sum_{k=1}^{N} q_k \phi_k{}^h$$

for some weights $q_1, \ldots, q_N$. The latter are determined by (22), i.e.,

$$\sum_{k=1}^{N} q_k a(\phi_k^h, \phi_j^h) = -a(g_h, \phi_j^h)$$

for $j = 1, \ldots, N$, or what is the same,

$$Kq = g,$$

where the (j, k) entry of K is

$$a(\phi_k^h, \phi_j^h) = \int_\Omega [-\nabla\phi_k^h \cdot \nabla\phi_j^h + \omega^2\phi_k^h\phi_j^h] + \int_{\Gamma_\infty} T(\phi_k^h)\, \phi_j^h. \tag{23}$$

The function $T(\phi_k^h)$ comes from the generalized radiation condition (12); thus

$$T(\phi_k^h) = \sum_{l=1}^{\infty} \sigma_l a_l(\phi_k^h)\, H_+^\nabla(\sigma_l r_\infty) \sin \omega_l z, \tag{24}$$

where

$$a_l(\phi_k^h) = \frac{2}{H_\infty H_+(\sigma_l r_\infty)} \int_0^{H_\infty} dz\, (\phi_k^h(r_\infty, z) \sin \omega_l z. \tag{25}$$

Observe that the exact evaluation of the term

$$\int_{\Gamma_\infty} T(\phi_k^h)\, \phi_j^h$$

in (23) requires the summation of the series

$$\sum \alpha_l^k \alpha_l^j \beta_l,$$

where

$$\alpha_l^k = \frac{2}{H_\infty} \int_0^{H_\infty} dz\, \phi_k^h(r_\infty, z) \sin \omega_l z$$

and

$$\beta_l = \sigma_l \frac{H_+^\nabla(\sigma_l r_\infty)}{H_+(\sigma_l r_\infty)}.$$

Since the functions ϕ_k^h in the basis for S_0^h have square integrable gradients and vanish at $z = 0$, it follows that

$$\alpha_l^k = O(l^{-2}).$$

Since $\beta_l = O(l)$, the terms in the series are of order $O(l^{-3})$. This is a sufficiently fast convergence so the series can be summed without appreciable computational effort.

However, for the special configuration used here it is necessary to retain only those terms in the series expansion of T which correspond to values of the index l at which

$$\omega_l < \omega \qquad (\text{say } \omega_1, \ldots, \omega_N < \omega < \omega_{N+1}).$$

Indeed, let us denote this truncated version of T by T_A and, for purposes of illustration, solve the problem for the case of the flat top, i.e., $z = H(r) \equiv H_\infty$ with $\partial\phi/\partial n = T_A(\phi)$ on Γ_∞. In doing so we find that the error is given by

$$\sum_{k=N+1}^{\infty} b_k \Delta_k(r) \sin(\omega_k z),$$

where the b_k's are the Fourier coefficients of the data given on $r = r_0$ and $\Delta_k(r)$ is given by

$$\Delta_k(r) = \frac{H_+^\nabla(\sigma_k r_\infty)}{H_+(\sigma_k r_0)} \frac{H_+(\sigma_k r_0)\, H_-(\sigma_k r) - H_+(\sigma_k r)\, H_-(\sigma_k r_0)}{H_-^\nabla(\sigma_k r_\infty)\, H_+(\sigma_k r_0) - H_+^\nabla(\sigma_k r_\infty)\, H_-(\sigma_k r_0)}.$$

Using the asymptotic expansions (4) and similar expansions for the derivatives of the Hankel functions, we may conclude that

$$|\Delta_k(r)| \leqslant c e^{-|\sigma_k|(r_\infty - r_0)}$$

for r_∞ sufficiently large and for $r_0 \leqslant r \leqslant r_\infty$ (we have used the fact that $\sigma_k = i\,|\,\sigma_k\,|$ for $k \geqslant N+1$). Thus, for a fixed number of terms in T_A, the error decays exponentially as a function of r_∞.

4. Numerical Experiments

The numerical results reported in this section used

$$H(r) = H_\infty \left\{1 + \frac{\alpha}{r+1} \sin \beta r\right\} \tag{26}$$

for the top boundary and

$$g(z) = \sin \omega_1 z. \tag{27}$$

The space S^h consists of piecewise linear functions on a distorted triangular grid whose nodal points are at

$$r_{jl} = r_0 + jh(r_\infty - r_0),$$
$$z_{jl} = lhH_\infty(r_{jl}),$$

for $j, l = 1, \ldots, 1/h$.

The first set of experiments deals with a flat top where $\alpha = 0$.

$$r_0 = 1,\ r_\infty = 10,\ H_\infty = 1. \tag{28}$$

(The exact solution ϕ in this case is

$$\phi(r, z) = H_+(\sigma_1 r) \sin \omega_1 z. \tag{29}$$

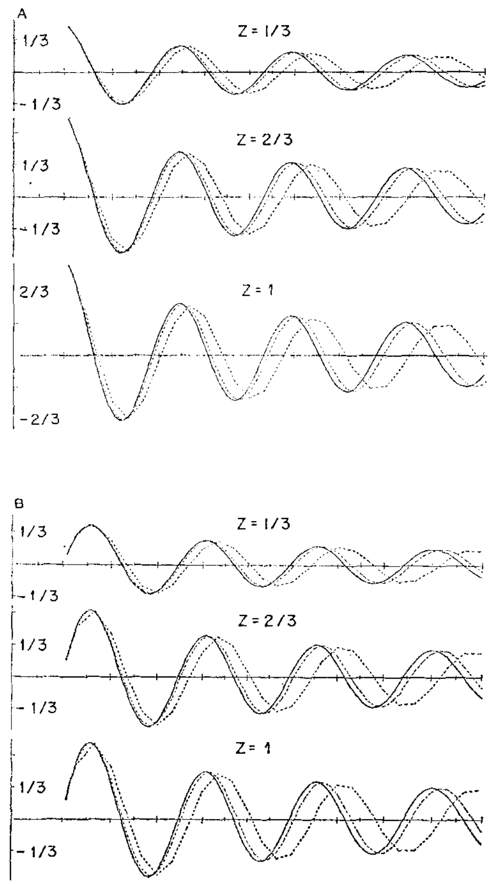

FIG. 1. (A) $\omega = 3$, $\alpha = 0$, real part of ϕ_h. (B) $\omega = 3$, $\alpha = 0$, imaginary part of ϕ_h.

Figure 1 contains a plot of the approximate solution ϕ_h as a function of r various values of z and

$$\omega h = 1, \tfrac{1}{2}, \tfrac{1}{4}, \tag{30}$$

with $\omega = 3$. The second order convergence is qualitatively clear from this figure, and was confirmed quantitatively by the authors by direct computation of the error $\varphi - \varphi_h$.

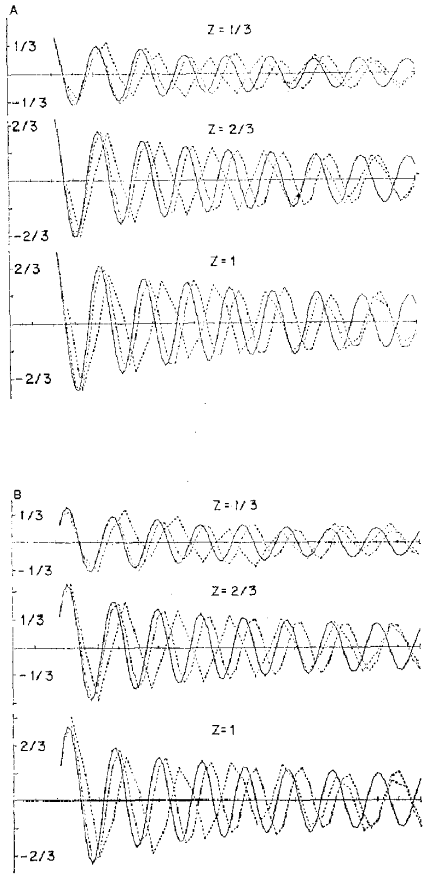

FIG. 2. (A) $\omega = 6$, $\alpha = 0$, real part of ϕ_h. (B) $\omega = 6$, $\alpha = 0$, imaginary part of ϕ_h.

Observe that for the parameters (28)

$$\omega_k = \left(\frac{2k-1}{2}\right)\pi;$$

hence

$$\omega_1 < \omega < \omega_2$$

for $\omega = 3$. Accordingly only one term in the power series expansion for $T(\cdot)$ was used in the computation of φ_h.

Figure 2 contains the analogous results for $\omega = 6$ where

$$\omega_2 < \omega < \omega_3,$$

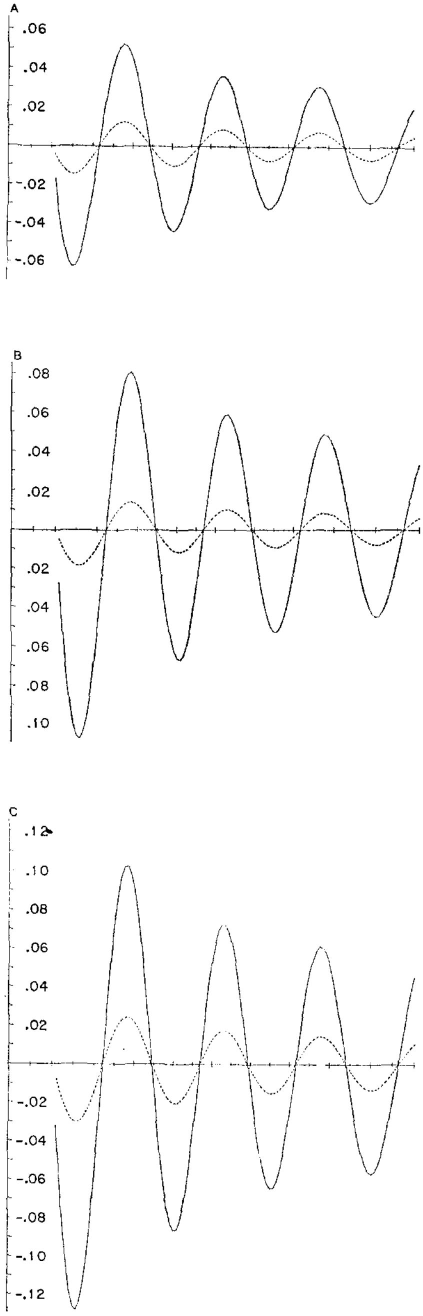

FIG. 3. (A) The difference (31), $\omega = 3$, $\alpha = 0$, $z = \frac{1}{3}$, solid line is real part. (B) The difference (31), $\omega = 3$, $\alpha = 0$, $z = \frac{2}{3}$, solid line is real part. (C) The difference (31), $\omega = 3$, $\alpha = 0$, $z = 1$, solid line is real part.

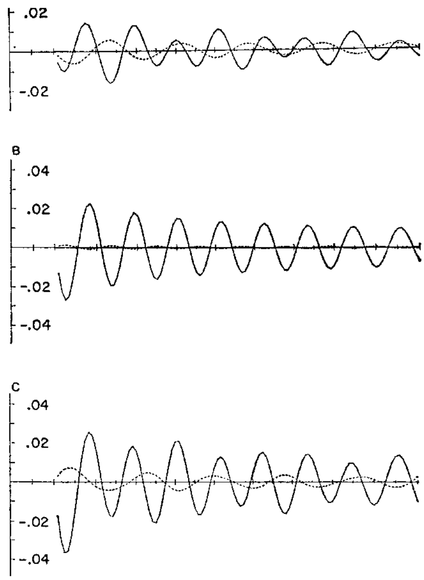

FIG. 4. (A) The difference (31), $\omega = 6$, $\alpha = 0$, $z = \frac{1}{3}$, solid line is real part. (B) The difference (31), $\omega = 6$, $\alpha = 0$, $z = \frac{2}{3}$, solid line is real part. (C) The difference (31), $\omega = 6$, $\alpha = 0$, $z = 1$ solid line is real part.

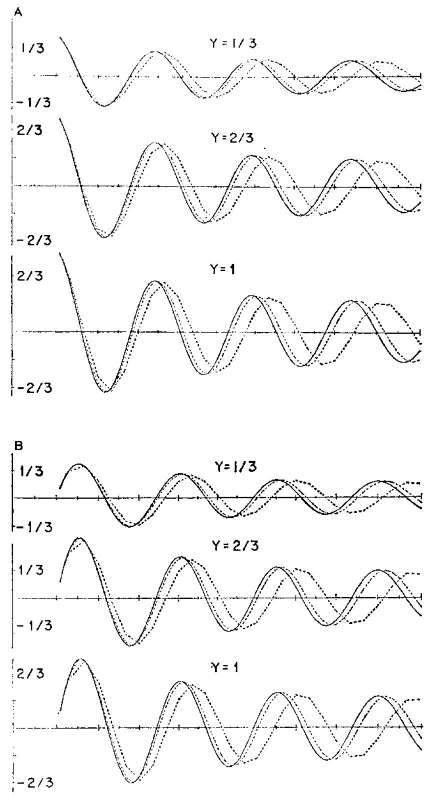

FIG. 5. (A) $\omega = 3$, curved top, real part of ϕ_h. (B) $\omega = 3$, curved top, imaginary part of ϕ_h.

except that

$$\omega h = 2, 1, \tfrac{1}{2}.$$

Two terms were used in $T(\cdot)$ for this ω in the computation of φ_h.

Figures 3 and 4 contain plots of the difference

$$\hat{\varphi} - \phi_h \tag{31}$$

between φ_h and the solution $\hat{\varphi}_h$ with the classical radiation boundary condition (3) for $\omega = 3$ and $\omega = 6$. The difference represents wall reflections in $\hat{\varphi}_h$ and is less severe for $\omega = 6$ than for $\omega = 3$. The reason for this was pointed out in Section 2 where it was noted that the generalized radiation condition approached the classical radiation condition as $\omega \to \infty$.

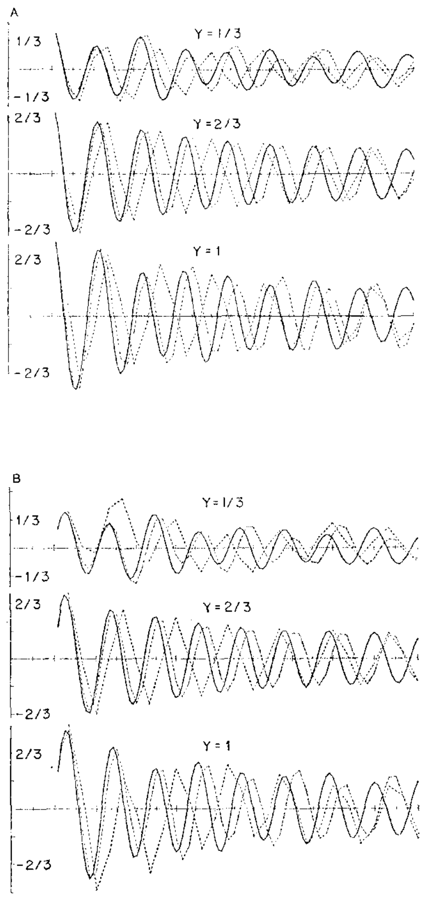

FIG. 6. (A) $\omega = 6$, curved top, real part of ϕ_h. (B) $\omega = 6$, curved top, imaginary part of ϕ_h

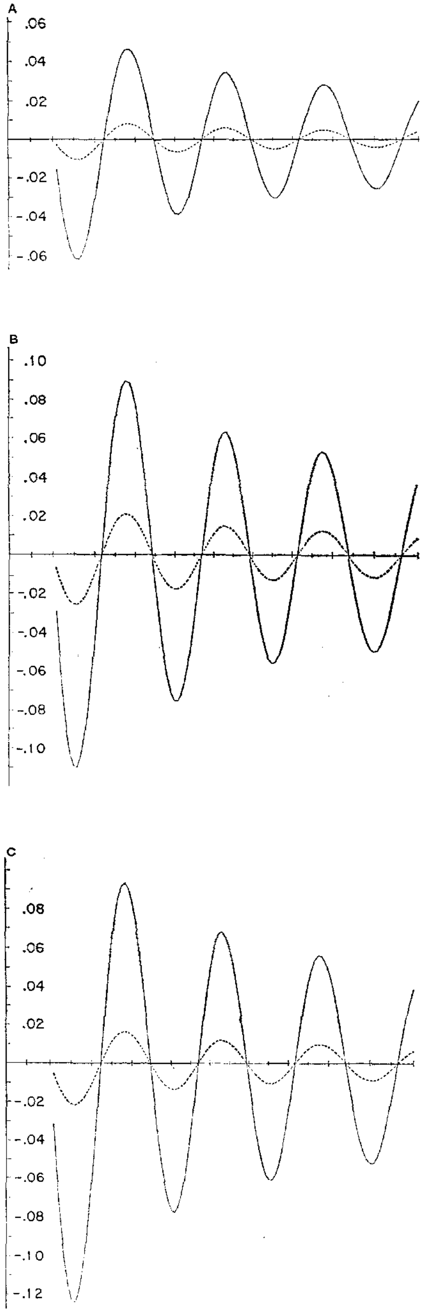

FIG. 7. (A) The difference (31), $\omega = 3$, curved top, $y = \frac{1}{3}$, solid line is real part. (B) The difference (31), $\omega = 3$, curved top, $y = \frac{2}{3}$, solid line is real part. (C) The difference (31), $\omega = 3$, curved top, $y = 1$, solid line is real part.

The next set of experiments deals with a curved top $z = H(z)$ with H given by (26) with

$$\alpha = .25, \beta = 1.26. \tag{32}$$

Figure 5 contains plots of φ_h as a function of r for various values of

$$y = z/H(r)$$

for $\omega = 3$ and ωh given by (30). Although the exact solution is not known in this case, the figures indicate second order convergence. The analogous results for $\omega = 6$ are plotted in Fig. 6, and Fig. 7 and 8 contain plots of the difference $\hat{\varphi}_h - \varphi_h$.

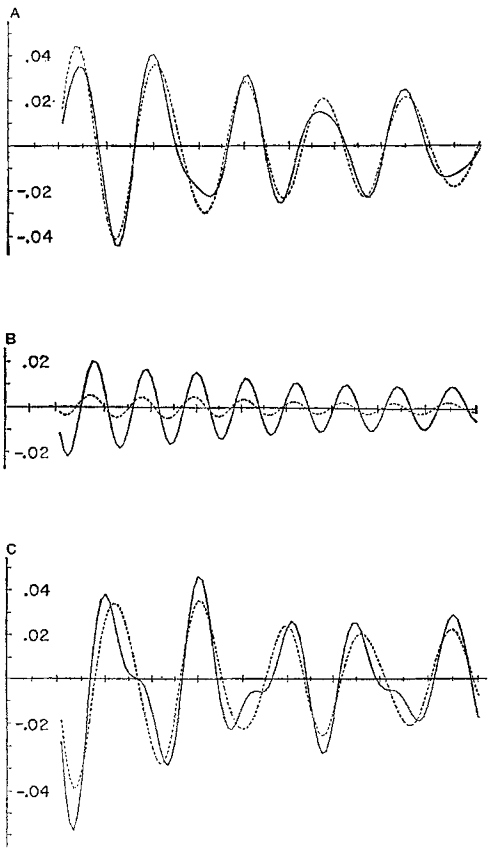

FIG. 8. (A) The difference (31), $\omega = 6$, curved top, $y = \frac{1}{3}$, solid line is real part. (B) The difference (31), $\omega = 6$, curved top, $y - \frac{2}{3}$, solid line is real part. (C) The difference (31), $\omega = 6$, curved top, $y = 1$, solid line is real part.

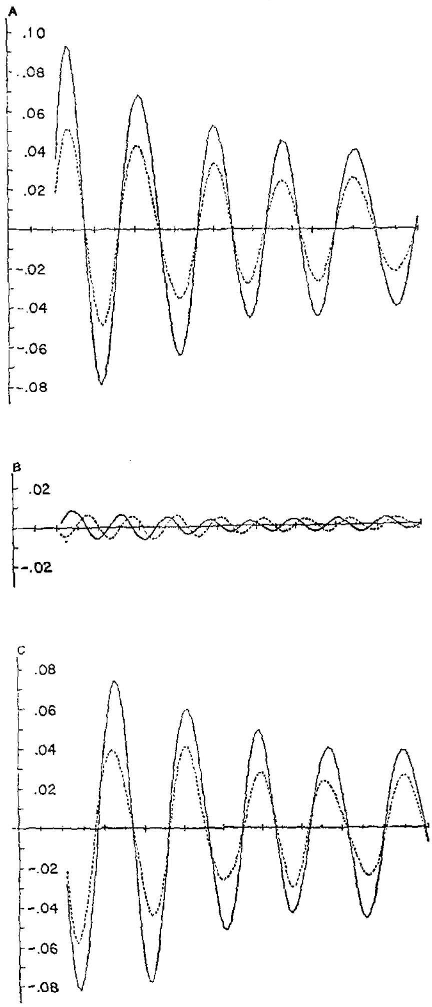

FIG. 9. (A) The difference (33), $\omega = 6$, curved top, $y = \frac{1}{3}$, solid line is real part. (B) The difference (33), $\omega = 6$, curved top, $y = \frac{2}{3}$, solid line is real part. (C) The difference (33), $\omega = 6$, curved top, $y = 1$, solid line is real part.

In the final figure we illustrate the importance of having the correct number of terms in $T(\cdot)$. In particular, Fig. 9 contains a plot of the difference

$$\phi_h^{(1)} - \phi_h^{(2)} \tag{33}$$

for $\omega = 6$ and a curved top, where $\varphi_h^{(l)}$ is the approximate solution with l terms in $T(\cdot)$. For $\omega = 6$ one term is not sufficient since $\omega_2 < \omega < \omega_3$ requires that two terms should be used in $T(\cdot)$. We also computed

$$\phi_h^{(1)} - \phi_h^{(3)}$$

and there was no detectable difference, which reflects the fact that these extra terms are exponentially small.

References

1. G. Chertock, "Integral Equations Methods in Sound Radiation and Scattering from Arbitrary Surfaces," Naval Ship Research and Development Center, Report 3538, 1971.
2. D. Greenspan and P. Werner, *Arch. Rational Mech. Anal.* **23** (1966), 288.
3. S. P. Marin, Thesis, in preparation.
4. B. Engquist and A. Majda, *Math.* **31** (July 1977).
5. L. M. Brekhovskikh, "Waves in Layered Media," Academic Press, 1960.